Investment Decisions Under Uncertainty: The "Irreversibility Effect"

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What is meant here by an "irreversible decision?" A decision is considered irreversible if it significantly reduces for a long time the variety of choices that would be possible in the future.

Suppose for instance that we in 1974 must decide whether the cathedral of Notre-Dame is to be preserved or to be demolished and replaced by a parking lot. A decision to preserve Notre-Dame is not irreversible since, if adopted in 1974, it leaves open the possibility of further choices between Notre-Dame and the parking lot (or other alternatives). However, if the parking lot is built in 1974, we will never again be in a position to choose to keep Notre-Dame; this certainly is an irreversible decision.

After ploughing his field, a farmer considers what kind of crop to plant now. Whatever decision he adopts is not irreversible, since he will later have to decide what to plant again in this field; the set of possibilities offered to him will not have been modified by the decision previously taken. On the other hand, if he decides to hew down a forest of full-grown oaks and bring more land into cultivation, he makes an irreversible decision. Generally, it is the different degrees of irreversibility associated with various possible decisions which are of interest: to build a power station where coal can be burned as well as oil is a "less irreversible" decision than to build an equally powerful station where only oil can be used as a fuel.

Our concern with irreversibilities[1] is actually related to the following problem: a new circumferential highway is now being planned around Paris as a direct connection between the various suburbs located ten kilometers beyond the city limits. It may cut through public forests (Versailles, Malmaison, etc.), ancient royal estates, that form a first green belt west of Paris. Will they be spared or will they be partially destroyed? The "Ministère de l'Equipement," responsible for highway planning, and the "Ministère de l'Agriculture," responsible for the management of public forests, are involved in a cost-benefit analysis on this point.

According to a rule systematically adopted at the "Ministère de l'Equipement," every random return or cost appearing in the problem is replaced by its expected value before application of any decision criterion (see, for example, G. Dreyfus). The initial random problem is thus replaced by an associated riskless problem, i.e., a (supposedly) "equivalent certainty case" in the sense of Herbert Simon and Henri Theil. Our approach here[2] is to show that irreversibility makes it impossible to draw the conclusions of Simon and Theil even if all the other assumptions of their models—including the quadratic payoff function—are satisfied.[3] Irreversibility even prevents the use of the concept Edmond Malinvaud calls "first-order certainty equivalence."

In fact using the information structure

* Centre National de la Recherche Scientifique and Ecole Polytechnique, Paris. This research has been supported by the C.N.R.S. under contract A.T.P. no. 62990 1. I am grateful to J. H. Drèze, G. M. Heal, A. P. Kirman, and E. Malinvaud, and an unknown referee for helpful comments, but of course, retain entire responsiblity for errors. Special thanks are due to A. C. Duyck for his help in computing the numerical examples.

[1] For other examples of irreversibilities, see P. W. Barkley and D. W. Seckler, pp. 149–55, and Anthony Fisher, John Krutilla, and Charles Cicchetti.

[2] In Kenneth Arrow and Fisher, the authors are making a similar point in a different framework.

[3] In the Appendix, a counterexample is presented which shows that, if we introduce irreversibility in Theil's model without changing any other of his assumptions, his result is no longer valid, i.e., the certainty equivalence does not hold, even for the first period.

defined in the next paragraph I will prove that, by replacing the initial random problem, even a risk-neutral[4] decision maker facing a binary alternative is led to adopt an irreversible decision more often than he should. Of course it is not surprising that, in general, replacing random variables by their expectations will not lead to the appropriate decisions. Our point is that this replacement will here, systematically and unduly, favor irreversible decisions, for example, destroying the forests and building the highway. In the following representative case that we numerically explore, the size of this "irreversibility effect" appears fairly important.

I. Uncertainty and Improvement of the Available Information

Let T denote the number of periods to be considered in the model; period t starts at time $t-1$ and ends at time t. At time t, $t=0, \ldots, T-1$, two situations are possible:

1) A choice actually can be made between two decisions; then $v_t=0$ means that no irreversible decision is made at time t, whereas $v_t=1$ means that an irreversible decision is made at time t. 2) No choice actually remains open at time t; then we will write $v_t=1$. Hence we have: if t is such that $v_t=0$, then for every τ such that $\tau<t$, $v_\tau=0$; if t is such that $v_t=1$, then for every τ such that $\tau \geq t$, $v_\tau=1$.

Let us now express the fact that, as time passes, the decision maker gets more and better information about the state of the world, hence about the returns which constitute the consequences of his decisions. Let S_t denote the information held by the decision maker at time t; this information is valuable to him in so far as it concerns not only past and present returns—present meaning here "at time t"—but also future returns. At any time t different from the initial one, i.e., $t>0$, there can be several possible states $S_{t,e}$, $e=1, \ldots, N(t)$, of the information available to the decision maker at time t; the net discounted return at time t, i.e., from the t-th period, associated with $S_{t,e}$ will be denoted $R_{t,e}$. The information available at time t in state $S_{t,e}$ about the returns at times later than t is specified by means of transition probabilities from this state $S_{t,e}$ to the various possible states $S_{t+1,f}$, $f=1, \ldots, N(t+1)$, i.e., by means of conditional probabilities $p_{t+1,f}^{t,e}$ of the states $S_{t+1,f}$ given $S_{t,e}$; these transition probabilities may depend on the sequence $\{v_0, \ldots, v_t\}$ of decisions taken up to time t.

In the terms of Malinvaud, what we have defined is an "information structure" which simultaneously is "fixed" and "has memory." Malinvaud's information structure is defined in the following manner: he considers that at any time t, the states $S_{t,e}$ of the information available at t constitute a partition of the space—which can be identified with S_0—of the states of the world; to guarantee that his information structure has memory, he makes the assumption that the partition S_{t+1} is as least as fine as the partition S_t. Our information structure is more general than his in so far as he doesn't allow the partition S_t to depend on the sequence of decisions $\{v_0, \ldots, v_{t-1}\}$; moreover we are considering coverings and not only partitions.

I shall now consider a problem of sequential decision under irreversibility and uncertainty, where uncertainty is measured by the above information structure. To the initial random problem we will associate a riskless problem where, for every period and every sequence of decisions affecting this period, the random returns from this period given this sequence of decisions are replaced by their expected value calculated at time 0, i.e., at the time the decision maker solves his problem in order to decide what must be done immediately. We will see that a risk-neutral decision maker having solved the associated riskless problem more often adopts an immediate[5] irreversible decision

[4] Risk neutrality is unessential but simplifies matters; for a more general presentation allowing for any attitude towards risk, see my 1973 paper, which also examines choices with more than two alternatives.

[5] We are not interested in decisions which are not immediate, i.e., which would be made now and applied after a delay: indeed, after a delay another decision problem emerges, with another "time 0," requiring an immediate effective decision to which Proposition 1 eventually applies.

than does the same decision maker who has solved the initial random problem. This is an immediate consequence of Proposition 1.

PROPOSITION 1: *Consider a sequential decision problem under uncertainty, where irreversible decisions may be made and where the information structure is fixed and has memory: suppose the decision maker is risk neutral; consider the associated riskless problem. If the solution of the associated problem doesn't imply an immediate irreversible decision, then the solution of the initial problem doesn't either; but it may happen that the solution of the associated problem implies an immediate irreversible decision, whereas the solution of the initial problem does not.*

PROOF: In the initial problem, let $Q^*_{t,e}(v_0, \ldots, v_{t-1})$ denote the maximum possible conditional expected value of future returns given the sequence $\{v_0, \ldots, v_{t-1}\}$ of decisions; "conditional" refers to the state $S_{t,e}$ of the information available at time t and "future" means times $t+1, \ldots, T$. The principle of optimality[6] that must be followed by a decision maker solving the initial problem thus reads:

$$(1)\quad Q^*_{t,e}(v_0, \ldots, v_{t-1}) = \max_{\{v_t \text{ compatible with } v_0, \ldots, v_{t-1}\}} E_{t, v_0, \ldots, v_t} \cdot [R_{t+1}(v_0, \ldots, v_t) + Q^*_{t+1}(v_0, \ldots, v_t)] = \max_{\{\text{idem}\}} \sum_f p^{t,e}_{t+1,f}(v_0, \ldots, v_t) \cdot (R_{t+1,f}(v_0, \ldots, v_t) + Q^*_{t+1,f}(v_0, \ldots, v_t))$$

In the associated riskless problem, let $\overline{Q}^*_t(v_0, \ldots, v_{t-1})$ denote the maximum possible value of future (sure) returns given the sequence $\{v_0, \ldots, v_{t-1}\}$; the principle of optimality that must be followed by a decision maker solving the associated riskless problem thus reads:

$$(2)\quad \overline{Q}^*_t(v_0, \ldots, v_{t-1}) = \max_{\{\text{idem}\}} (\overline{R}_{t+1}(v_0, \ldots, v_t) + \overline{Q}^*_{t+1}(v_0, \ldots, v_t))$$

where

$$(3)\quad \overline{R}_{t+1}(v_0, \ldots, v_t) = E_{v_0, \ldots, v_t}[R_{t+1}(v_0, \ldots, v_t)] = \sum_f p_{t+1,f}(v_0, \ldots, v_t) \cdot R_{t+1,f}(v_0, \ldots, v_t)$$

with

$$\forall t = 0, \ldots, T-1,\ P_{t+1,f} = \sum_e P_{t,e} \cdot P^{t,e}_{t+1,f}$$

We will first show that $\forall t = 0, \ldots, T-1$,

$$(4)\quad E_{v_0, \ldots, v_{t-1}}[Q^*_t(v_0, \ldots, v_{t-1})] \geq \overline{Q}^*_t(v_0, \ldots, v_{t-1})$$

This result is trivial when $1 \in \{v_0, \ldots, v_{t-1}\}$; furthermore equality then holds. Suppose now $1 \notin \{v_0, \ldots, v_{t-1}\}$; then

$$(5)\quad Q^*_{t,e}(v_0, \ldots, v_{t-1}) = \max_{v_t \in \{0,1\}} E_{t, v_0, \ldots, v_t}[R_{t+1}(v_0, \ldots, v_t) + Q^*_{t+1}(v_0, \ldots, v_t)]$$

hence, $\forall v_t \in \{0, 1\}$,

$$(6)\quad E_{v_0, \ldots, v_{t-1}}[Q^*_{t,e}(v_0, \ldots, v_{t-1})] \geq \overline{R}_{t+1}(v_0, \ldots, v_t) + E_{v_0, \ldots, v_t}[Q^*_{t+1}(v_0, \ldots, v_t)]$$

If $v_t = 1$, we now have

$$(7)\quad E_{v_0, \ldots, v_t}[Q^*_{t+1}(v_0, \ldots, v_t)] = \overline{Q}^*_{t+1}(v_0, \ldots, v_t)$$

It is thus enough to show that

$$(8)\quad E_{v_0, \ldots, v_{t-1}, 0}[Q^*_{t+1}(v_0, \ldots, v_{t-1}, 0)] \geq \overline{Q}^*_{t+1}(v_0, \ldots, v_{t-1}, 0)$$

As both members are zero for $t = T-1$, it is trivially true for $t = T-1$; it then results from

[6] About principles of optimality in stochastic dynamic programming see Arrow, note 4, pp. 524–25, and R. Bellman, pp. 199–200 and 205–09.

a recursive argument that it is also true for any $t \in \{1, \ldots, T-1\}$.

Hence, for $t=1$, we have

$$E_{v_0=1}[Q_1^*(v_0 = 1)] = \overline{Q}_1^*(v_0 = 1) \tag{9}$$

$$E_{v_0=0}[Q_1^*(v_0 = 0)] \geq \overline{Q}_1^*(v_0 = 0) \tag{10}$$

In the initial random problem, we choose $v_0=1$ if

$$\begin{aligned} W_1 &= E_{v_0=1}[R_1(v_0 = 1) + Q_1^*(v_0 = 1)] \\ &= \overline{R}_1(v_0 = 1) + E_{v_0=1}[Q_1^*(v_0 = 1)] \end{aligned} \tag{11}$$

is greater than

$$\begin{aligned} W_0 &= E_{v_0=0}[R_1(v_0 = 0) + Q_1^*(v_0 = 0)] \\ &= \overline{R}_1(v_0 = 0) + E_{v_0=0}[Q_1^*(v_0 = 0)] \end{aligned} \tag{12}$$

In the associated riskless problem, we choose $v_0=1$ if

$$\overline{W}_1 = \overline{R}_1(v_0 = 1) + \overline{Q}_1^*(v_0 = 1) \tag{13}$$

is greater than

$$\overline{W}_0 = \overline{R}_1(v_0 = 0) + \overline{Q}_1^*(v_0 = 0) \tag{14}$$

From (9), $W_1 = \overline{W}_1$; from (10), $W_0 \geq \overline{W}_0$.

For a case where $W_0 > W_1 = \overline{W}_1 > \overline{W}_0$, see the Appendix where it suffices to make:

$$v_t = x_{t+1},\ x_{t+1} \in \{0, 1\}, \qquad t = 0, 1 \tag{15}$$

$$S_0 = \{\omega_e \mid e = 1, 2\}, \tag{16}$$

$$\text{with } \begin{cases} y_1 = y_2 = -6 \text{ in the state } \omega_1 \\ y_1 = y_2 = +6 \text{ in the state } \omega_2 \end{cases}$$

$$S_{1,e} = \{\omega_e\}, \qquad e = 1, 2 \tag{17}$$

$$S_{2,f} = \{\omega_f\}, \qquad f = 1, 2 \tag{18}$$

$$\overset{0}{p}_{1,e} = \tfrac{1}{2}, \qquad e = 1, 2 \tag{19}$$

$$\overset{1,e}{p}_{2,f} = \delta_{ef}, \qquad e = 1, 2, \quad f = 1, 2 \tag{20}$$

$$R_{t,1} = (x_t + 2)(4 - x_t - 6), \quad t = 1, 2 \tag{21}$$

$$R_{t,2} = (x_t + 2)(4 - x_t + 6), \quad t = 1, 2 \tag{22}$$

$$\overline{R}_t = (x_t + 2)(4 - x_t), \qquad t = 1, 2 \tag{23}$$

Solving the initial random problem, the decision maker is led to adopt $v_0 = x_1 = 0$, whereas, by replacing this problem by the associated riskless problem, he is led to adopt $v_0 = x_1 = 1$.

It would be most interesting to have an idea of how large the irreversibility effect can be in a realistic case. With this aim in mind we have designed the following simulation. At time one $N(1) = 3^2$ different states of the available information are possible; they are denoted $S_{1,e(1)}$ where

$$e(1) = (e_I(1), e_R(1)) \tag{24}$$

$$e_I(1) \in \{1 - \beta, 1, 1 + \beta\} \tag{25}$$

$$e_R(1) \in \{1 - \gamma, 1, 1 + \gamma\} \tag{26}$$

furthermore for every $e(1)$ we have

$$\overset{0}{p}_{1,e(1)} = 1/9 \tag{27}$$

$$R_{1,e(1)}(v_0 = 1) = \frac{e_I(1)}{1 + \sigma} \tag{28}$$

$$R_{1,e(1)}(v_0 = 0) = F \cdot \frac{e_R(1)}{1 + \sigma} \tag{29}$$

This means that if an irreversible decision is made at time 0 (i.e., $v_0 = 1$) the undiscounted return at time one will be either $1-\beta$, or 1, or $1+\beta$, with probabilities 1/3, respectively; if no irreversible decision is made at time 0 (i.e., $v_0=0$), the undiscounted return at time one will be either $F\cdot(1-\gamma)$, or F, or $F\cdot(1+\gamma)$, with probabilities 1/3, respectively; the returns from $v_0=0$ are stochastically independent of the returns from $v_0=1$ (this is consistent with $N(1)=3^2$).

At time two $N(2)=6^2$ different states of the available information are possible; they are denoted $S_{2,e(2)}$ where

$$e(2) = (e_I(2), e_R(2)) \tag{30}$$

$$e_I(2) \in \{(1-\beta)^a(1+\beta)^b \mid 0 \leq a+b \leq 2\} \tag{31}$$

$$e_R(2) \in \{(1-\gamma)^c(1+\gamma)^d \mid 0 \leq c+d \leq 2\} \tag{32}$$

a, b, c, and d being nonnegative integers; furthermore for every $e(1)$ and every $e(2)$ we have (33)–(35). This means that, if for example $e_I(1) = 1-\beta$ and $e_R(1) = 1+\gamma$, there are only nine attainable states at time two, which are $(e_I(2), e_R(2))$ with

$$(33)\qquad p_{2,e(2)}^{1,e(1)}\begin{cases} = 1/9 & \text{if } \begin{cases} e_I(2) \in \{e_I(1)\cdot(1-\beta),\ e_I(1),\ e_I(1)\cdot(1+\beta)\} \\ \text{and} \\ e_R(2) \in \{e_R(1)\cdot(1-\gamma),\ e_R(1),\ e_R(1)\cdot(1+\gamma)\} \end{cases} \\ = 0 & \text{otherwise} \end{cases}$$

$$(34)\qquad R_{2,e(2)}(v_1 = 1) = \frac{e_I(2)}{(1+\sigma)^2}$$

$$(35)\qquad R_{2,e(2)}(v_1 = 0) = F\cdot\frac{e_R(2)}{(1+\sigma)^2}$$

$$(36)\qquad e_I(2) \in \{(1-\beta)\cdot(1-\beta),\ (1-\beta),\ (1-\beta)\cdot(1+\beta)\}$$

$$(37)\qquad e_R(2) \in \{(1+\gamma)\cdot(1-\gamma),\ (1+\gamma),\ (1+\gamma)\cdot(1+\gamma)\}$$

If an irreversible decision has been made at time 0 or is made at time one (i.e., $v_1=1$), the undiscounted return at time two will be either $(1-\beta)\cdot(1+\beta)$, or $(1-\beta)$, or $(1-\beta)\cdot(1+\beta)$, with probabilities 1/3, respectively. If no irreversible decision is taken by time one (including, i.e., $v_1=0$), the undiscounted return at time two will be either $(1+\gamma)\cdot(1-\gamma)$, or $(1+\gamma)$, or $(1+\gamma)\cdot(1+\gamma)$, with probabilities 1/3, respectively. Again the respective consequences, of $v_1=1$ and of $v_1=0$ are stochastically independent.

At time t,

$$N(t) = \left(\frac{(t+1)\cdot(t+2)}{2}\right)^2$$

different states of the available information are possible; they are denoted $S_{t,e(t)}$ where

$$(38)\qquad e(t) = (e_I(t),\ e_R(t))$$

$$(39)\qquad e_I(t) \in \{(1-\beta)^a(1+\beta)^b \mid 0 \le a+b \le t\}$$

$$(40)\qquad e_R(t) \in \{(1-\gamma)^c(1+\gamma)^d \mid 0 \le c+d \le t\}$$

a, b, c, and d being nonnegative integers; furthermore for every $e(t-1)$ and every $e(t)$ we have (41)–(43).

For $T=10$ and different values of σ, β, γ, and F, we then have the following results: if $\sigma=0.05$ then $W_1=7.72$ and W_0 is given by Table 1. If $\sigma=0.10$ then $W_1=6.14$ and W_0 is given by Table 2.

As we have always chosen $\beta=\gamma$, the difference between W_0 and W_1 is a consequence only of: 1) the difference between 1 and F, and 2) the fact that W_0 results from a reversible decision, while W_1 results from an irreversible one.

Hence, for given σ, β, and γ (with $\beta=\gamma$), we may choose as a measure of the irreversibility effect the difference between 1 and that value of F, denoted $F_{IE}(\sigma, \beta, \gamma)$, which equalizes W_0 and W_1; for example

$$(44)\qquad F_{IE}(0.10, 0.10, 0.10) = 0.87$$

$$(41)\qquad p_{t,e(t)}^{t-1,e(t-1)}\begin{cases} = 1/9 & \text{if } \begin{cases} e_I(t) \in \{e_I(t-1)\cdot(1-\beta),\ e_I(t-1),\ e_I(t-1)\cdot(1+\beta)\} \\ \text{and} \\ e_R(t) \in \{e_R(t-1)\cdot(1-\gamma),\ e_R(t-1),\ e_R(t-1\ \cdot(1+\gamma)\} \end{cases} \\ = 0 & \text{otherwise} \end{cases}$$

$$(42)\qquad R_{t,e(t)}(v_{t-1} = 1) = \frac{e_I(t)}{(1+\sigma)^t}$$

$$(43)\qquad R_{t,e(t)}(v_{t-1} = 0) = F\cdot\frac{e_R(t)}{(1+\sigma)^t}$$

TABLE 1

	$F=1$	$F=0.95$	$F=0.90$	$F=0.85$	$F=0.80$
$\beta=\gamma=0.05$	7.99	7.78	7.65	7.58	7.53
$\beta=\gamma=0.10$	8.26	8.03	7.84	7.69	7.57

TABLE 2

	$F=1$	$F=0.95$	$F=0.90$	$F=0.85$	$F=0.80$
$\beta=\gamma=0.05$	6.34	6.18	6.07	6.00	5.96
$\beta=\gamma=0.10$	6.54	6.36	6.20	6.08	5.99

as

$$(45)\quad W_0(\sigma=0.10, \beta=0.10, \gamma=0.10, F=0.87) = W_1(\sigma=0.10)=6.14$$

As might be expected, the irreversibility effect decreases with increasing σ (increasing σ meaning that future returns, hence uncertainty on future returns, are considered as less important) and increases with increasing $\beta=\gamma$ (i.e., with increasing uncertainty on future returns). As $\sigma=0.10$ is a usual choice for a discount rate and as $\beta=\gamma=0.10$ is a moderate estimation of uncertainty for many public investment programs,[7] 13 percent seems to give a good idea of the degree of magnitude of the irreversibility effect in a case where this effect can be considered as moderate.

APPENDIX

I shall use Theil's notations, assuming that T=2 (number of periods), $m=1$ (number of "instruments" or "controlled variables"), and $n=1$ (number of "noncontrolled variables" or "result variables"). The noncontrolled variable y_t, t=1, 2, which can be identified with Theil's "random element" s(t), is subject to the following simultaneous probability distribution:

$$(A1)\quad Pr\,[y_1=y_2=-6]=Pr\,[y_1=y_2=6]=\tfrac{1}{2}$$

The "welfare function," i.e., the function whose expected value the decision maker wishes to maximize, reads

$$(A2)\quad \omega(x, y) = (x_1+2)(4-x_1+y_1) + (x_2+2)(4-x_2+y_2)$$

Irreversibility on the instrument x appears in the fact that the set where the decision maker will be allowed to choose a value for x_2 depends on the value he previously chooses for x_1, in the following way:

$$(A3)\quad x_1 \leq x_2$$

Let us solve this decision problem, taking into account that at the beginning of the second period, the decision maker will hold better information about the state of the world that obtains than he does at the beginning of the first period. Suppose he chooses $x_1 \in R$ at the beginning of the first period. If $y_1=y_2=-6$ obtains, he will then have to solve the following problem at the beginning of the second period:

$$(A4)\quad \max\,(x_2+2)(4-x_2-6)$$

under condition (A3), whose solution is

$$(A5)\quad \begin{cases} x_2 = -2 & \text{if } x_1 \leq -2 \\ x_2 = x_1 & \text{if } x_1 \geq -2 \end{cases}$$

If, on the other hand, $y_1=y_2=6$ obtains, the decision maker will then have to solve the following problem at the beginning of the second period:

$$(A6)\quad \max\,(x_2+2)\,(4-x_2+6)$$

under condition (A3), whose solution is

$$(A7)\quad \begin{cases} x_2 = 4 & \text{if } x_1 \leq 4 \\ x_2 = x_1 & \text{if } x_1 \geq 4 \end{cases}$$

Hence, to choose the appropriate value of x_1 at the beginning of the first period, the decision maker has to solve:

$$(A8)\quad \max\,G(x_1)$$

distinguishing three cases to explicit $G(x_1)$:

1) $x_1 \leq -2$; then

$$(A9)\quad G(x_1)=\tfrac{1}{2}[(x_1+2)(4-x_1-6)+0] + \tfrac{1}{2}[(x_1+2)(4-x_1+6) + (4+2)(4-4+6)]$$

[7] We consider cases where T=10; hence a single period in our scheme will in general correspond to more than one year; even more than ten years in the "highway versus forests" case, in which case $\beta=\gamma=0.10$ is not a moderate but a very low estimation of uncertainty.

hence, on $[-\infty, -2]$, $G(x_1)$ reaches a unique maximum $G(-2)=18$.

2) $-2 \leq x_1 \leq 4$; then

$$\text{(A10)} \quad G(x_1) = \tfrac{1}{2}[(x_1+2)(4-x_1-6) + (x_1+2)(4-x_1-6)] + \tfrac{1}{2}[(x_1+2)(4-x_1+6) + (4+2)(4-4+6)]$$

hence, on $[-2, 4]$, $G(x_1)$ reaches a unique maximum $G(0)=24$.

3) $4 \leq x_1$; then

$$\text{(A11)} \quad G(x_1) = \tfrac{1}{2}[(x_1+2)(4-x_1-6) + (x_1+2)(4-x_1-6)] + \tfrac{1}{2}[(x_1+2)(4-x_1+6) + (x_1+2)(4-x_1+6)]$$

hence, on $[4, +\infty]$, $G(x_1)$ reaches a unique maximum $G(4)=0$.

In order to maximize $G(x_1)$, the decision maker therefore chooses $x_1=0$ as the value of the instrument for the first period.

Let us now examine the associated riskless problem, i.e., the "certainty case" in Theil's words. The decision maker must now maximize

$$\text{(A12)} \quad (x_1+2)(4-x_1) + (x_2+2)(4-x_2)$$

under condition (A3), which clearly implies the decision $x_1=1$ for the first period.

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